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Advanced Linear Algebra (MA 409)
Problem Sheet - 5

Bases and Dimension

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1. Label the following statements as true or false.
   (a) The zero vector space has no basis.
   (b) Every vector space that is generated by a finite set has a basis.
   (c) Every vector space has a finite basis.
   (d) A vector space cannot have more than one basis.
   (e) If a vector space has a finite basis, then the number of vectors in every basis is the same.
   (f) The dimension of $P_n(F)$ is n.
   (g) The dimension of $M_{m\times n}(F)$ is $m+n$.
   (h) Suppose that V is a finite-dimensional vector space, that S_1 is a linearly independent subset of V, and that S_2 is a subset of V that generates V. Then S_1 cannot contain more vectors than S_2.
   (i) If S generates the vector space V, then every vector in V can be written as a linear combination of vectors in S in only one way.
   (j) Every subspace of a finite-dimensional space is finite-dimensional.
   (k) If V is a vector space having dimension n, then V has exactly one subspace with dimension 0 and exactly one subspace with dimension n.
   (l) If V is a vector space having dimension n, and if S is a subset of V with n vectors, then S is linearly independent if and only if S spans V.

2. Determine which of the following sets are bases for $\mathbb{R}^3$.
   (a) $\{(1,2,-1),(1,0,2),(2,1,1)\}$
   (b) $\{(1,-3,-2),(-3,1,3),(-2,-10,-2)\}$

3. Determine which of the following sets are bases for $P_2(\mathbb{R})$.
   (a) $\{-1-x+2x^2, 2+x-2x^2, 1-2x+4x^2\}$
   (b) $\{1-2x-2x^2, -2+3x-x^2, 1-x+6x^2\}$
   (c) $\{1+2x-x^2, 4-2x+x^2, -1+18x-9x^2\}$

4. Do the polynomials x^3-2x^2+1, $4x^2-x+3$, and $3x-2$ generate $P_3(\mathbb{R})$? Justify your answer.

5. Is $\{(1,4,-6),(1,5,8),(2,1,1),(0,1,0)\}$ a linearly independent subset of $\mathbb{R}^3$? Justify your answer.

6. Give three different bases for F^2 and for $M_{2\times 2}(F)$.

7. The vectors $u_1 = (2, -3, 1)$, $u_2 = (1, 4, -2)$, $u_3 = (-8, 12, -4)$, $u_4 = (1, 37, -17)$, and $u_5 = (-3, -5, 8)$ generate $\mathbb{R}^3$. Find a subset of the set $\{u_1, u_2, u_3, u_4, u_5\}$ that is a basis for $\mathbb{R}^3$.

8. Let W denote the subspace of $\mathbb{R}^5$ consisting of all the vectors having coordinates that sum to zero. The vectors

$$\begin{aligned} u_1 &= (2, -3, 4, -5, 2), & u_2 &= (-6, 9, -12, 15, -6), \\ u_3 &= (3, -2, 7, -9, 1), & u_4 &= (2, -8, 2, -2, 6), \\ u_5 &= (-1, 1, 2, 1, -3), & u_6 &= (0, -3, -18, 9, 12), \\ u_7 &= (1, 0, -2, 3, -2), & u_8 &= (2, -1, 1, -9, 7) \end{aligned}$$

generate W. Find a subset of the set $\{u_1, u_2, \ldots, u_8\}$ that is a basis for W.

9. The vectors $u_1 = (1, 1, 1, 1)$, $u_2 = (0, 1, 1, 1)$, $u_3 = (0, 0, 1, 1)$, and $u_4 = (0, 0, 0, 1)$ form a basis for F^4. Find the unique representation of an arbitrary vector (a_1, a_2, a_3, a_4) in F^4 as a linear combination of u_1,u_2,u_3, and u_4.

10. In each part, use the Lagrange interpolation formula to construct the polynomial of smallest degree whose graph contains the following points.

    (a) $(-2, -6), (-1, 5), (1, 3)$
    (b) $(-4, 24), (1, 9), (3, 3)$
    (c) $(-2, 3), (-1, -6), (1, 0), (3, -2)$
    (d) $(-3, -30), (-2, 7), (0, 15), (1, 10)$

11. Let u and v be distinct vectors of a vector space V. Show that if $\{u, v\}$ is a basis for V and a and b are nonzero scalars, then both $\{u + v, au\}$ and $\{au, bv\}$ are also bases for V.

12. Let u, v, and w be distinct vectors of a vector space V. Show that if $\{u, v, w\}$ is a basis for V, then $\{u + v + w, v + w, w\}$ is also a basis for V.

13. The set of solutions to the system of linear equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - 3x_2 + x_3 &= 0 \end{aligned}$$

is a subspace of $\mathbb{R}^3$. Find a basis for this subspace.

14. Find bases for the following subspaces of F^5:

$$W_1 = \left\{ (a_1, a_2, a_3, a_4, a_5) \in F^5 : a_1 - a_3 - a_4 = 0 \right\}$$

and

$$W_2 = \left\{ (a_1, a_2, a_3, a_4, a_5) \in F^5 : a_2 = a_3 = a_4 \text{ and } a_1 + a_5 = 0 \right\}.$$

What are the dimensions of W_1 and W_2?

15. The set of all $n \times n$ matrices having trace equal to zero is a subspace W of $M_{n\times n}(F)$. Find a basis for W. What is the dimension of W?

16. The set of all upper triangular $n \times n$ matrices is a subspace W of $M_{n\times n}(F)$. Find a basis for W. What is the dimension of W?

17. The set of all skew-symmetric $n \times n$ matrices is a subspace W of $M_{n\times n}(F)$. Find a basis for W. What is the dimension of W?

18. Find a basis for the vector space V consisting of all sequences $\{a_n\}$ in a field F that have only a finite number of nonzero terms a_n.

19. Let W_1 and W_2 be subspaces of a finite-dimensional vector space V. Determine necessary and sufficient conditions on W_1 and W_2 so that $\dim(W_1 \cap W_2) = \dim(W_1)$.

20. Let $v_1, v_2, \ldots, v_k$, v be vectors in a vector space V, and define $W_1 = span(\{v_1, v_2, \ldots, v_k\})$, and $W_2 = span(\{v_1, v_2, \ldots, v_k, v\})$.

    (a) Find necessary and sufficient conditions on V such that $\dim(W_1) = \dim(W_2)$.

    (b) State and prove a relationship involving $\dim(W_1)$ and $\dim(W_2)$ in the case that $\dim(W_1) \neq \dim(W_2)$.

21. Let $f(x)$ be a polynomial of degree n in $P_n(\mathbb{R})$. Prove that for any $g(x) \in P_n(\mathbb{R})$ there exist scalars $c_0, c_1, \ldots, c_n$ such that

$$g(x) = c_0 f(x) + c_1 f'(x) + c_2 f''(x) + \cdots + c_n f^{(n)}(x),$$

    where $f^{(n)}(x)$ denotes the nth derivative of $f(x)$.

22. Let V and W be vector spaces over a field F. Let

$$Z = \{(v, w) : v \in V \text{ and } w \in W\}.$$

    Prove that Z is a vector space over F with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \text{ and } c(v_1, w_1) = (cv_1, cw_1).$$

    If V and W are vector spaces over F of dimensions m and n, determine the dimension of Z.

23. For a fixed $a \in \mathbb{R}$, determine the dimension of the subspace of $P_n(\mathbb{R})$ defined by $\{f \in P_n(\mathbb{R}) : f(a) = 0\}$.

24. Let W_1 denote the set of all polynomials $f(x)$ in $P(F)$ such that in the representation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

    we have $a_i = 0$ whenever i is even. Likewise let W_2 denote the set of all polynomials $g(x)$ in $P(F)$ such that in the representation

$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0,$$

    we have $b_i = 0$ whenever i is odd. We proved that $P(F) = W_1 \oplus W_2$. Determine the dimensions of the subspaces $W_1 \cap P_n(F)$ and $W_2 \cap P_n(F)$.

25. Let V be a finite-dimensional vector space over $\mathbb{C}$ with dimension n. Prove that if V is now regarded as a vector space over $\mathbb{R}$, then $\dim V = 2n$.

26. (a) Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $W_1 + W_2$ is finite-dimensional, and $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$. [Hint : Start with a basis $\{u_1, u_2, \ldots, u_k\}$ for $W_1 \cap W_2$ and extend this set to a basis

$$\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m\}$$

for W_1 and to a basis

$$\{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_p\}$$

for W_2.]

(b) Let W_1 and W_2 be finite-dimensional subspaces of a vector space V, and let $V = W_1 + W_2$. Deduce that V is the direct sum of W_1 and W_2 if and only if $\dim(V) = \dim(W_1) + \dim(W_2)$.

27. Let

$$V = M_{2\times 2}(F), W_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in V : a, b, c \in F \right\},$$

and

$$W_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \in V : a, b \in F \right\}.$$

Prove that W_1 and W_2 are subspaces of V, and find the dimensions of W_1, W_2, $W_1 + W_2$, and $W_1 \cap W_2$.

28. Let W_1 and W_2 be subspaces of a vector space V having dimensions m and n, respectively, where $m \geq n$.

(a) Prove that $\dim(W_1 \cap W_2) \leq n$.

(b) Prove that $\dim(W_1 + W_2) \leq m + n$.

29. (a) Find an example of subspaces W_1 and W_2 of $\mathbb{R}^3$ with dimensions m and n, where $m > n > 0$, such that $\dim(W_1 \cap W_2) = n$.

(b) Find an example of subspaces W_1 and W_2 of $\mathbb{R}^3$ with dimensions m and n, where $m > n > 0$, such that $\dim(W_1 + W_2) = m + n$.

(c) Find an example of subspaces W_1 and W_2 of $\mathbb{R}^3$ with dimensions m and n, where $m \geq n$, such that both $\dim(W_1 \cap W_2) < n$ and $\dim(W_1 + W_2) < m + n$.

30. (a) Let W_1 and W_2 be subspaces of a vector space V such that $V = W_1 \oplus W_2$. If β_1 and β_2 are bases for W_1 and W_2, respectively, show that $\beta_1 \cap \beta_2 = \varnothing$ and $\beta_1 \cup \beta_2$ is a basis for V.

(b) Conversely, let β_1 and β_2 be disjoint bases for subspaces W_1 and W_2, respectively, of a vector space V. Prove that if $\beta_1 \cup \beta_2$ is a basis for V, then $V = W_1 \oplus W_2$.

31. (a) Prove that if W_1 is any subspace of a finite-dimensional vector space V, then there exists a subspace W_2 of V such that $V = W_1 \oplus W_2$.

(b) Let $V = \mathbb{R}^2$ and $W_1 = \{(a_1, 0) : a_1 \in R\}$. Give examples of two different subspaces W_2 and W_2' such that $V = W_1 \oplus W_2$ and $V = W_1 \oplus W_2'$.

32. Let W be a subspace of a finite-dimensional vector space V, and consider the basis $\{u_1, u_2, \ldots, u_k\}$ for W. Let $\{u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_n\}$ be an extension of this basis to a basis for V.

    (a) Prove that $\{u_{k+1} + W, u_{k+2} + W, \ldots, u_n + W\}$ is a basis for V/W.

    (b) Derive a formula relating $\dim(V)$, $\dim(W)$, and $\dim(V/W)$.

33. Let V be the set of all 2×2 matrices A with complex entries which satisfy $A_{11} + A_{22} = 0$,

    (a) Show that V is a vector space over the field of real numbers, with the usual operations of matrix addition and multiplication of a matrix by a scalar.

    (b) Find a basis for this vector space.

    (c) Let W be the set of all matrices A in V such that $A_{21} = -\overline{A_{12}}$ (the bar denotes complex conjugation). Prove that W is a subspace of V and find a basis for W.

34. Let V be the set of real numbers. Regard V as a vector space over the field of rational numbers, with the usual operations. Prove that this vector space is not finite-dimensional.

35. Show that the set of all ordered triplets (x_1, x_2, x_3) of real numbers such that

$$\frac{x_1}{3} = \frac{x_2}{4} = \frac{x_3}{2}$$

    forms a real vector space, where the operations $+$ and . are as in $\mathbb{R}^3$. Find a basis and hence the dimension of the vector space.

36. Let S be a subset of a vector space V and $A \subseteq S$. Then the following statements are equivalent :

    (a) A is a minimal set with the property $Sp(A) \supseteq S$;

    (b) Every element of S can be expressed uniquely as a linear combination from A;

    (c) $Sp(A) \supseteq S$ and A is linearly independent;

    (d) A is a maximal linearly independent subset of S.

37. Find a basis of the vector space $P(\Omega)$ (with the operations defined in the problem sheet "Vector Spaces"), when Ω is an arbitrary non-empty finite set.

38. Find all the bases of the following subspaces.

    (a) For any non-empty subset A of Ω, $\{\varnothing, A\}$ is a subspace.

    (b) For any distinct non-empty subsets A and B of Ω, $\{\varnothing, A, B, A \triangle B\}$ is another subspace.

39. If $\{x_1, x_2, \ldots, x_k\}$ is a basis of a subspace S, show that

    (a) $\{\alpha x_1, x_2, \ldots, x_k\}$ is a basis of S iff $\alpha \neq 0$.

    (b) $\{x_1 + \beta x_2, x_2, \ldots, x_k\}$ is a basis of S for any scalar β,

    (c) $\{x_1 + \beta x_2, \alpha x_1 + x_2, x_3, \ldots, x_k\}$ is a basis of S iff $\alpha\beta \neq 1$.

40. If a subspace S of $\mathbb{R}^n$ has a basis $\{x_1, x_2, \ldots, x_k\}$ such that all components of x_1 are strictly positive, show that S has a basis B such that all components of each vector in B are strictly positive.

41. Let $A \subseteq V$. If one vector in $Sp(A)$ can be expressed uniquely as a linear combination from A then show that A is linearly independent and, so, is a basis of $Sp(A)$.

42. Show that a vector space V over F has a unique basis iff either "$d(V) = 0$" or "$d(V) = 1$ and $|F| = 2$".

43. Prove or disprove: if A, B and C are pair-wise disjoint subsets of V such that $A \cup B$ and $A \cup C$ are bases of V, then $Sp(B) = Sp(C)$.

44. Prove or disprove: if B is a basis of V and S is a subspace of V then B contains a basis of S.

45. Consider the basis

$$B = \Big\{(1,-1,0,0,0), (1,0,-1,0,0), (1,0,0,-1,0), (1,0,0,0,-1)\Big\}$$

of the subspace

$$S = \Big\{(u_1, u_2, u_3, u_4, u_5) : u_1 + u_2 + \ldots + u_5 = 0\Big\}$$

of $\mathbb{R}^5$. Using B, extend the linearly independent subset $\{x_1, x_2\}$ of S to a basis of S, where $x_1 = (1,0,0,2,-3)$ and $x_2 = (1,1,0,4,-6)$.

46. Extend $A = \{(1,1,\ldots,1)\}$ to a basis of $\mathbb{R}^n$.

47. Let $x_1, x_2, \ldots, x_n$ be fixed distinct real numbers.

    (a) Show that $\ell_1(t), \ell_2(t), \ldots, \ell_n(t)$ form a basis of $P_n(\mathbb{R})$, where $\ell_i(t) = \prod_{j \neq i}(t - x_j)$. This basis leads to what is known as **Lagrange's interpolation formula**. If $f(t) \in P_n(\mathbb{R})$ is written as $\sum_{i=1}^{n} \alpha_i \ell_i(t)$, show that $\alpha_i = f(x_i)/\ell_i(x_i)$.

    (b) Show that $\psi_1(t), \psi_2(t), \ldots, \psi_n(t)$ form a basis of $P_n(\mathbb{R})$, where $\psi_1(t) = 1$ and $\psi_i(t) = \prod_{j=1}^{i-1}(t - x_j)$ for $i = 2, \ldots, n$. This basis leads to what is known as **Newton's divided difference formula**.

48. Find a basis of each of the following subspaces of $\mathbb{R}^4$. Also express S_3 in the form

$$\left\{(x_1, x_2, x_3, x_4) : \frac{x_1}{a_1} = \frac{x_2}{a_2} = \frac{x_3}{a_3} = \frac{x_4}{a_4}\right\}$$

    (a) $S_1 = \{(x_1, x_2, x_3, x_4) : x_1 - 2x_3 + x_4 = 0\}$,

    (b) $S_2 = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 - x_3 = 0, x_2 + 2x_3 - x_4 = 0, 2x_1 + 3x_2 - x_4 = 0\}$,

    (c) $S_3 = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 - x_3 = 0, x_1 + x_2 + 2x_3 + x_4 = 0, x_1 - 3x_2 - x_3 + 2x_4 = 0\}$.

49. Let $B = \{x_1, x_2, \cdots, x_k\}$ be a basis of S and $x = \alpha_1 x_1 + \ldots + \alpha_k x_k \notin B$. Obtain a necessary and sufficient condition for $(B \cup \{x\}) - \{x_i\}$ to be a basis of S.

50. Show that the subspaces of continuous functions and differentiable functions are not finite-dimensional.

51. Let F be a finite field with q elements and V an n-dimensional vector space over F.

    (a) Show that $|V| = q^n$.

    (b) Show (using the Theorem : The vectors $x_1, x_2, \ldots, x_k$ are linearly dependent iff x_j belongs to the span of $x_1, x_2, \ldots, x_{j-1}$ for some j such that $\leq j \leq k$) that the number of ordered k-tuples $(x_1, x_2, \ldots, x_k)$ such that $x_1, x_2, \ldots, x_k$ are linearly independent vectors in V, is

$$(q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{k-1})$$

(c) Show that the number of distinct (unordered) bases of V is

$$(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})/n!$$

(d) Show that the number of k-dimensional subspaces of V is

$$\frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})}=\frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}$$

This number is usually denoted by $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

(e) Show that the number of ℓ-dimensional subspaces of V containing a given k-dimensional subspace is $\begin{bmatrix} n-k \\ \ell-k \end{bmatrix}_q$.

52. If F is a subfield of a finite field G, prove that the number of elements in G is a power of the number of elements in F.

53. Let F be a subfield of a field G and let $x_l, x_2, \ldots, x_k \in F^n$. Show that $x_l, x_2, \ldots, x_k$ are linearly independent in F^n over F iff they are linearly independent in G^n over G.

[Hint : first consider the case $k = n$.]